RECIROCAL SUMS OF l-TH POWER OF GENERALIZED BINARY SEQUENCES WITH INDICES

EMRAH KILIÇ AND NURETTIN IRMAK

Abstract. Recently in [5], the author considered certain reciprocal sums of general second order recurrence \{W_n\}. In this paper, we generalize the results of Xi and we give some new results for the reciprocal sums of l-th power of general second order recurrence \{W_{kn}\} for arbitrary positive integer k.

1. Introduction

Let \(a, b, P\) and \(Q\) integers such that \(PQ \neq 0\) and \(P^2 - 4Q \neq 0\). Define the sequence \{\(W_n\)\} as follows: for \(n > 1\)

\[ W_n = PW_{n-1} - QW_{n-2} \]  \hspace{1cm} (1.1)

where \(W_0 = a, W_1 = b\). The sequence \{\(W_n\)\} and its some properties have been studied by several authors. Horadam [3] gave the Binet form of \{\(W_n\)\} as shown :

\[ W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta} \]

where \(\alpha, \beta = (P \pm \sqrt{P^2 - 4Q})/2\), \(A = b - \beta A, B = b - \alpha A\). We denote \(W_n\) by \(W_n(a, b, P, Q)\). As important special cases, denote \(W_n(0, 1, P, Q)\) and \(W_n(2, P, P, Q)\) by \(U_n\) and \(V_n\), respectively.

Some authors have studied the both finite and infinite reciprocal sums of terms of certain sequences. In [1], the authors derived

\[ \sum_{n=1}^m \frac{Q^n}{W_n W_{n+k}} = U_m \sum_{n=1}^k \frac{Q^n}{W_n W_{n+m}} \quad \text{and} \quad \sum_{n=1}^\infty \frac{Q^n}{W_n W_{n+k}} = \frac{1}{ABU_k} \left( \sum_{n=1}^k \frac{W_{n+1}}{W_n} - k\alpha \right) \]  \hspace{1cm} (1.2)

where \(P > 0\) and \(k, m\) nonnegative integers. For the case \(Q = -1\), the identities in (1.2) are obtained by Good [2]. Regarding taking \(l\)-th powers of terms in the sums, the author [5] generalized the results of [1, 2]. For

2000 Mathematics Subject Classification. 11B39, 11B37.

Key words and phrases. Second order recurrence, reciprocal sums, power.
example, he derived the following infinitive reciprocal sums:

\[
\sum_{n=k}^{\infty} \frac{Q^n}{W_n W_{n+m}} \sum_{i=0}^{l-1} \left\{ (W_{k+1} - W_k \beta) Q^{n-k} \alpha^m - (W_{k+1} - W_k \alpha) \times \beta^{2(n-k)+m-1-i} \left[ (W_{k+2} - W_k \beta) Q^{n-k} \beta^m - (W_{k+1} - W_k \alpha) \beta^{2(n-k)+m} \right] \right\} \\
= \frac{(r^2 - 4Q)^{1/2} Q^k}{(\alpha^m - \beta^m)(W_{k+1} - W_k \beta)} \sum_{i=0}^{m-1} \beta^{ri} W_{k+i}.
\]

In [5], the authors gave the general results including the earlier results by taking \( l \)-th powers of terms in the reciprocal sums. In this paper, we generalize the result of [5] regarding reciprocal sums of \( l \)-th powers of the terms with indices.

2. The Main Results

In this section we consider both finite and infinite reciprocal sums of products of \( r \)-consecutive terms of the sequence \( \{W_n\} \). Clearly we will consider the finite and infinite reciprocal sums of \( \{W_n\}_{n=0}^{\infty} \) for arbitrary positive integer \( r \). For later use and the readers convenience, we have the following result from [4]:

**Lemma 1.** Let \( W_n \) be the \( n \)th term of sequence \( \{W_n\} \). Then for \( n, r > 0 \),

\[
W_{rn} = V_r W_{r(n-1)} - Q^r W_{r(n-2)}
\]

where \( V_n \) and \( Q \) be as before.

For our purpose, we use the generating function of the sequence \( \{W_{rn}\} \). We give our first result.

**Theorem 1.** Let \( P > 0 \),

\[
\sum_{n=k}^{\infty} \frac{Q^n}{W_n W_{n+m}} \sum_{i=0}^{l-1} \left\{ (W_{r(k+1)} - W_{rk} \beta^r) Q^{r(n-k)} \alpha^r - (W_{r(k+1)} - W_{rk} \alpha^r) \beta^{2r(n-k)+r} \right\} \\
= \frac{(\alpha^m - \beta^m)^{-1} Q^k}{(W_{r(k+1)} - W_{rk} \beta^r)} \left[ \frac{1}{W_{rk}} - \frac{\beta^{r(k+1-k)}}{W_{r(k+1)}} \right].
\]

**Proof.** For ease write and arbitrary \( r > 0 \), let \( f(x) = \sum_{n=k}^{\infty} W_{rn} x^n \). Thus consider

\[
f(x) - W_{rk} x^k - W_{r(k+1)} x^{k+1} = V_r x (f(x) - W_{rk} x^k) - Q^r x^2 f(x).
\]

Hence

\[
f(x) = x^k \frac{W_{rk} + x(W_{r(k+1)} - (\alpha^r + \beta^r) W_{rk})}{1 - (\alpha^r + \beta^r)x + (\alpha^r + \beta^r)x^2}.
\]
Thus of decomposed into partial fractions:

\[ f(x) = \frac{x^k}{\alpha^r - \beta^r} \left( \frac{W_{r(k+1)} - \beta^r W_{rk}}{1 - \alpha^r x} - \frac{W_{r(k+1)} - \alpha^r W_{rk}}{1 - \beta^r x} \right). \]

From the coefficients of \( x^n \) in both sides above equations, we get

\[ W_{rn} = \frac{W_{r(k+1)} - \beta^r W_{rk}}{\alpha^r - \beta^r} \alpha^r(n-k) - \frac{W_{r(k+1)} - \alpha^r W_{rk}}{\alpha^r - \beta^r} \beta^r(n-k). \]

Let \( T_{rn} = \frac{\beta^{r(n-k)}}{W_{rn}} \). If we compute the difference of two consecutive terms of \( \{T_{rk}\} \), we obtain

\[
T_{rn} - T_{(r+1)} = \frac{(\beta^{r(n-k)}W_{r(n+1)})-(\beta^{r(n+1-k)}W_{rn})}{W_{rn}W_{r(n+1)}}
\]

\[ = \frac{1}{W_{rn}W_{r(n+1)}}(\alpha^r - \beta^r) \left\{ ([W_{r(k+1)} - W_{rk}\beta^r])Q^r(n-k)\alpha^r 
- ([W_{r(k+1)} - W_{rk}\alpha^r])\beta^{2r(n-k)+r} \right\} 
- ([W_{r(k+1)} - W_{rk}\beta^r]Q^r(n-k)\beta^r - ([W_{r(k+1)} - W_{rk}\alpha^r])\beta^{2r(n-k)+r}) 
- (W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r}, \}

Thus

\[
T_{rn} - T_{r(n+1)} = \frac{(W_{r(k+1)} - W_{rk}\beta^r)Q^r(n-k)}{W_{rn}W_{r(n+1)}(\alpha^r - \beta^r)} \times \sum_{i=0}^{l-1} \left\{ ([W_{r(k+1)} - W_{rk}\beta^r])Q^r(n-k)\alpha^r - ([W_{r(k+1)} - W_{rk}\alpha^r])\beta^{2r(n-k)+r} \right\}^{-1-i} 
- ([W_{r(k+1)} - W_{rk}\beta^r]Q^r(n-k)\beta^r - ([W_{r(k+1)} - W_{rk}\alpha^r])\beta^{2r(n-k)+r})^{-1-i}. \]

Then we obtain

\[
\sum_{n=k}^{l} \frac{\alpha^r - \beta^r}{W_{rn}W_{r(n+1)}} \sum_{i=0}^{l-1} \left\{ ([W_{r(k+1)} - W_{rk}\beta^r])Q^r(n-k)\alpha^r 
- ([W_{r(k+1)} - W_{rk}\alpha^r])\beta^{2r(n-k)+r} \right\}^{-1-i} 
- ([W_{r(k+1)} - W_{rk}\beta^r]Q^r(n-k)\beta^r - ([W_{r(k+1)} - W_{rk}\alpha^r])\beta^{2r(n-k)+r})^{-1-i} 
= \frac{(\alpha^r - \beta^r)(l+1-k)}{W_{r(k+1)} - W_{rk}\beta^r} \sum_{n=k}^{l} \left( T_{rn} - T_{r(n+1)} \right) 
= \frac{(\alpha^r - \beta^r)(l+1-k)}{W_{r(k+1)} - W_{rk}\beta^r} \left[ \frac{1}{W_{rk} - \beta^{r(t+1-k)}} \right].
\]

409
Thus the proof is complete.

For example, when $l = 1$ in (2.2), we have the following result for $P > 0$

$$\sum_{n=k}^{t} \frac{Q^n}{W_{kn} W_{k(n+1)}} = \frac{Q^k}{(W_{r(k+1)} - W_{rk})} \left( \frac{1}{W_{rk}} - \frac{\beta^{r(t+1-k)}}{W_{r(t+1)}} \right).$$

The case $r = 1$ in the above result can be found in [5]. As a numerical example, if we take $W_n (0, 1, 1, 1) = F_n$, then

$$\sum_{n=1}^{t} \frac{1}{F_{2n} F_{2(n+1)}} = \beta^2 - \frac{\beta^{2t+2}}{F_{2(t+1)}}.$$

**Theorem 2.** For $P > 0$,

$$\sum_{n=k}^{\infty} \frac{Q^n}{W_{kn} W_{k(n+1)}} \sum_{i=0}^{t-1} \{ (W_{r(k+1)} - W_{rk}) Q^{r(n-k)} \alpha^r
-(W_{r(k+1)} - W_{rk}) \beta^{2r(n-k)+r} \}^{i-1}$$

$$\times [W_{r(k+1)} - W_{rk}) Q^{r(n-k)} \beta^r - (W_{r(k+1)} - W_{rk}) \beta^{2r(n-k)+r}]$$

$$= \frac{(\alpha^r - \beta^r)^{t}}{W_{rk}(W_{r(k+1)} - W_{rk})^{-1} Q^k}.$$

**Proof.** From Theorem 1,

$$\sum_{n=k}^{\infty} \frac{Q^n}{W_{kn} W_{k(n+1)}} \sum_{i=0}^{t-1} \{ (W_{r(k+1)} - W_{rk}) Q^{r(n-k)} \alpha^r
-(W_{r(k+1)} - W_{rk}) \beta^{2r(n-k)+r} \}^{i-1}$$

$$\times [W_{r(k+1)} - W_{rk}) Q^{r(n-k)} \beta^r - (W_{r(k+1)} - W_{rk}) \beta^{2r(n-k)+r}]$$

$$= \frac{(\alpha^r - \beta^r)^{t}}{W_{rk}(W_{r(k+1)} - W_{rk})^{-1} Q^k} \left( \frac{1}{W_{rk}} - \lim_{t \to \infty} \frac{\beta^{r(t+1-k)}}{W_{r(t+1)}} \right)$$

$$= \frac{(\alpha^r - \beta^r)^{t}}{W_{rk}(W_{r(k+1)} - W_{rk})^{-1} Q^k}.$$

Since

$$\left| \frac{\alpha}{\beta} \right| > 1 \quad \text{for} \quad P > 0,$$

and

$$\lim_{t \to \infty} T_{rt}^t = \lim_{t \to \infty} \left( \frac{\beta^{r(t-k)}}{W_{rt}} \right)^t = \lim_{t \to \infty} \left[ \frac{W_{r(k+1)} - \beta^r W_{rk}}{\alpha^r - \beta^r} \left( \frac{\alpha}{\beta} \right)^{r(t-k)} - \frac{W_{r(k+1)} - \alpha^r W_{rk}}{\alpha^r - \beta^r} \right]^{-t} = 0. \quad (2.4)$$

The proof is complete.
RECIPROCAL SUMS OF $l$-TH POWER OF GENERALIZED BINARY SEQUENCES

If we take $l = 1$ in Theorem 2, we get

$$\sum_{n=k}^{\infty} \frac{Q^n_{r_{(n+1)}}}{W_{r_{(n+1)}}} = \frac{Q^k_{r_k}}{W_{r_k}(W_{r_{(k+1)}} - W_{r_{(k+1)}}^2)}.$$  

Theorem 3. For $P > 0$ and $m > p$,

$$\sum_{n=k}^{l} \frac{Q^{(n+p)}}{W_{r_{(n+p)}}^l} \sum_{i=0}^{l-1} \left\{ [(W_{r_{(k+1)}} - W_{r_{k}}^2) Q^{r(n+p-k)}]_{r(m-p)} \right.$$

$$- (W_{r_{(k+1)}} - W_{r_{k}}^2) \beta^{2r(n-k)+r(m+p)} \right)^{l-1-i}$$

$$\times \left\{ [(W_{r_{(k+1)}} - W_{r_{k}}^2) Q^{r(n+p-k)}]_{r(m-p)} - (W_{r_{(k+1)}} - W_{r_{k}}^2) \beta^{2r(n-k)+r(m+p)} \right\}$$

$$= \frac{U_{(k+1)}^l Q^k_{r_k}}{(W_{r_{(k+1)}} - W_{r_{k}}^2 U_{r_{(m-p)}}) W_{r_{(n+p)}}^l W_{r_{(n+m)}}^l} \sum_{i=0}^{l-1} \left\{ [(W_{r_{(k+1)}} - W_{r_{k}}^2) Q^{r(n+p-k)}]_{r(m-p)} \right.$$

$$- (W_{r_{(k+1)}} - W_{r_{k}}^2) \beta^{2r(n-k)+r(m+p)} \right)^{l-1-i}$$

$$\times \left\{ [(W_{r_{(k+1)}} - W_{r_{k}}^2) Q^{r(n+p-k)}]_{r(m-p)} - (W_{r_{(k+1)}} - W_{r_{k}}^2) \beta^{2r(n-k)+r(m+p)} \right\}.$$  

Proof. By (2.3), we write

$$T^l_{r(n+p)} - T^l_{r(n+m)} = \left( \frac{\beta^{r(n+p-k)}}{W_{r_{(n+p)}}^l} \right) - \left( \frac{\beta^{r(n+m-k)}}{W_{r_{(n+m)}}^l} \right)$$

$$= \left( \frac{\beta^{r(n+p-k)} W_{r_{(n+m)}}^l - \beta^{r(n+m-k)} W_{r_{(n+p)}}^l}{W_{r_{(n+p)}}^l W_{r_{(n+m)}}^l} \right)$$

$$= \left( \frac{\beta^{r(n+p-k)} W_{r_{(n+1)}}^l - \beta^{r(n+m-k)} W_{r_{(n+p)}}^l}{W_{r_{(n+p)}}^l W_{r_{(n+m)}}^l} \right)$$

$$\times \sum_{i=0}^{l-1} \left\{ [(W_{r_{(k+1)}} - W_{r_{k}}^2) Q^{r(n+p-k)}]_{r(m-p)} \right.$$

$$- (W_{r_{(k+1)}} - W_{r_{k}}^2) \beta^{2r(n-k)+r(m+p)} \right)^{l-1-i}$$

$$\times \left\{ [(W_{r_{(k+1)}} - W_{r_{k}}^2) Q^{r(n+p-k)}]_{r(m-p)} - (W_{r_{(k+1)}} - W_{r_{k}}^2) \beta^{2r(n-k)+r(m+p)} \right\}.$$

Thus

$$T^l_{r(n+p)} - T^l_{r(n+m)} = \left( \frac{\beta^{r(n+p-k)} W_{r_{(n+1)}}^l - \beta^{r(n+m-k)} W_{r_{(n+p)}}^l}{W_{r_{(n+p)}}^l W_{r_{(n+m)}}^l} \right)$$

$$\times \sum_{i=0}^{l-1} \left\{ [(W_{r_{(k+1)}} - W_{r_{k}}^2) Q^{r(n+p-k)}]_{r(m-p)} \right.$$

$$- (W_{r_{(k+1)}} - W_{r_{k}}^2) \beta^{2r(n-k)+r(m+p)} \right)^{l-1-i}$$

$$\times \left\{ [(W_{r_{(k+1)}} - W_{r_{k}}^2) Q^{r(n+p-k)}]_{r(m-p)} - (W_{r_{(k+1)}} - W_{r_{k}}^2) \beta^{2r(n-k)+r(m+p)} \right\}.$$
Theorem 4. Hence we have

\[
\begin{align*}
\sum_{n=k}^{\ell} \frac{Q^{r(n+p)}}{W_{r(n+p)}W_{r(n+m)}} \sum_{i=0}^{l-1} \{ & (W_{r(k+1)} - W_{rk \beta^r})Q^{r(n+p-k)}\alpha^{r(m-p)} \\
& - (W_{r(k+1)} - W_{rk \alpha^r})\beta^2r(n-k)+r(m+p)]^{l-1-i} \\
& \times (W_{r(k+1)} - W_{rk \beta^r})Q^{r(n+p-k)}\beta^{r(m-p)} \\
& - (W_{r(k+1)} - W_{rk \alpha^r})\beta^2r(n-k)+r(m+p)]^i \} \\
= & \frac{U_i^{r}Q^{rk}}{(W_{r(k+1)} - W_{rk \beta^r})U_{r(n-p)}} \sum_{n=k}^{\ell} \frac{U_i^{r}}{U_{r(n-p)}} \sum_{i=0}^{l-1} \left( \beta^{r_i} - \frac{\beta^{r_i(t+i+1-k)}}{W_{r(t+i+1)}} \right).
\end{align*}
\]

So we have the conclusion. □

As an example, when \( l = 1 \) in Theorem 3, one can obtain

\[
\sum_{n=k}^{\ell} \frac{Q^{r(n+p)}}{W_{r(n+p)}W_{r(n+m)}} = \frac{U_i^{r}Q^{rk}}{(W_{r(k+1)} - W_{rk \beta^r})U_{r(n-p)}} \sum_{i=0}^{m-1} \left( \beta^{r_i} - \frac{\beta^{r_i(t+i+1-k)}}{W_{r(t+i+1)}} \right).
\]

**Theorem 4.** For \( P > 0 \) and \( m > p \),

\[
\begin{align*}
\sum_{n=k}^{\ell} \frac{Q^{r(n+p)}}{W_{r(n+p)}W_{r(n+m)}} \sum_{i=0}^{l-1} \{ & (W_{r(k+1)} - W_{rk \beta^r})Q^{r(n+p-k)}\alpha^{r(m-p)} \\
& - (W_{r(k+1)} - W_{rk \alpha^r})\beta^2r(n-k)+r(m+p)]^{l-1-i} \\
& \times (W_{r(k+1)} - W_{rk \beta^r})Q^{r(n+p-k)}\beta^{r(m-p)} \\
& - (W_{r(k+1)} - W_{rk \alpha^r})\beta^2r(n-k)+r(m+p)]^i \} \\
= & \frac{U_i^{r}Q^{rk}}{(W_{r(k+1)} - W_{rk \beta^r})U_{r(n-p)}} \sum_{i=0}^{m-1} \left( \beta^{r_i} - \frac{\beta^{r_i(t+i+1-k)}}{W_{r(t+i+1)}} \right).
\end{align*}
\]

**Proof.** Considering (2.4) and (2.5), we have

\[
\begin{align*}
\sum_{n=k}^{\ell} \frac{Q^{r(n+p)}}{W_{r(n+p)}W_{r(n+m)}} \sum_{i=0}^{l-1} \{ & (W_{r(k+1)} - W_{rk \beta^r})Q^{r(n+p-k)}\alpha^{r(m-p)} \\
& - (W_{r(k+1)} - W_{rk \alpha^r})\beta^2r(n-k)+r(m+p)]^{l-1-i} \\
& \times (W_{r(k+1)} - W_{rk \beta^r})Q^{r(n+p-k)}\beta^{r(m-p)} \\
& - (W_{r(k+1)} - W_{rk \alpha^r})\beta^2r(n-k)+r(m+p)]^i \} \\
= & \frac{U_i^{r}Q^{rk}}{(W_{r(k+1)} - W_{rk \beta^r})U_{r(n-p)}} \sum_{i=0}^{m-1} \left( \beta^{r_i} - \frac{\beta^{r_i(t+i+1-k)}}{W_{r(t+i+1)}} \right).
\end{align*}
\]
Thus the proof is complete.

For example, by \( W_n (0, 1, 1, -1) = F_n \), from Theorem 4, we get
\[
\sum_{n=1}^{\infty} \frac{1}{F_{2n} F_{2n+4}} = \frac{8-3\sqrt{5}}{9}.
\]
Also for \( W_n (2, 1, 1, -1) = L_n \), we have
\[
\sum_{n=1}^{\infty} \frac{1}{F_{2n+2} F_{2n+6}} \left( \frac{(35+21\sqrt{5})}{2} - \left( 40 - 18\sqrt{5} \right) \beta^{4n} \right) = \frac{3953}{31792} - \frac{5287}{95299} \sqrt{5}
\]
and
\[
\sum_{n=1}^{t} \frac{1}{F_{4(n+3)} F_{4(n+6)}} = \left( \frac{1}{480} \sqrt{5} - \frac{1612592335}{3461453534698} \right) - \left( \frac{7}{1440} \sqrt{5} - \frac{1}{96} \right) \left( \frac{\beta^{4(t+3)}}{F_{4(t+4)}} + \frac{\beta^{4(t+4)}}{F_{4(t+5)}} + \frac{\beta^{4(t+5)}}{F_{4(t+6)}} \right)
\]
Also choosing by appropriate parameter, many special cases can be obtained.

References